Probabilistic Methods in Combinatorics

Solutions to Assignment 10

Problem 1. It is easy to prove that any graph with n vertices and maximum degree d contains an independent set of size at least $n/(d+1)$. The goal of this exercise is to show that at the price of decreasing the size of such a set by a constant factor we can guarantee that it has a certain structure.

Let $G = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of G into r pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geq 2ed$, where e is the basis of the natural logarithm. Show that there is an independent set of vertices $W \subseteq V$ that contains a vertex from each V_i.

Solution. By possibly removing some vertices from some of the sets V_i, we may assume that $|V_i| = \lceil 2ed \rceil$ for every $1 \leq i \leq r$. Let X be a random set, formed by picking, uniformly at random, exactly one vertex from each set V_i. For every edge uv, let A_{uv} be the event that X contains both u and v. Let i, j be such that $u \in V_i$ and $v \in V_j$. Then

$$\mathbb{P}[A_{uv}] = \begin{cases} 0 & i = j \\ \left(\frac{1}{\lceil 2ed \rceil}\right)^2 & i \neq j. \end{cases}$$

Moreover, the event A_{uv} is mutually independent of the collection of all events $A_{u'v'}$ but those events where $u'v'$ is incident with $V_i \cup V_j$. Note that the number of edges incident with $V_i \cup V_j$ is at most $2\lceil 2ed \rceil \cdot d$. Note that one of these edges is uv, so A_{uv} is mutually independent of a collection of all but at most $2d\lceil 2ed \rceil - 1$ events $A_{u'v'}$.

Write $p = \left(\frac{1}{\lceil 2ed \rceil}\right)^2$ and $D = 2d\lceil 2ed \rceil - 1$. Note that $e(D+1)p \leq 1$. Then, by the local lemma, we find that

$$\mathbb{P}\left[\bigcap_{uv \in E(G)} \overline{A_{uv}}\right] > 0.$$

It follows that there exists an independent set X that consists of exactly one vertex from each set V_i.

Problem 2. Show that for $d \geq 2$, every d-regular graph $G = (V, E)$ contains a set U such that for every vertex $v \in V$, the neighbourhood $N(v)$ of v satisfies $1 \leq |N(v) \cap U| \leq 50 \log d$.

Solution. Let $p = \frac{25 \log d}{d}$. Put each vertex in U with probability p, independently. (This is not valid when $p > 1$, but in that case $50 \log d > d$, so we can take $U = V$.) For each vertex v, let A_v be the event that either $N(v) \cap U = \emptyset$, or $|N(v) \cap U| \geq 50 \log d$. We shall show, using the local lemma, that with positive probability, none of the events A_v hold. This would imply the existence of a set U such that $1 \leq |N(v) \cap U| \leq 50 \log d$ for every vertex v, as required.

In order to verify that the conditions of the local lemma hold, we first upper-bound the probability $\mathbb{P}[A_v]$. Given v, write $X_v = |N(v) \cap U|$. Note that X_v has a binomial distribution $\mathrm{Bin}(d, p)$ with expectation $dp = 25 \log d$.

$$\mathbb{P}(A_v) \leq \mathbb{P}\Big[|X_v - \mathbb{E}[X_v]| \geq \mathbb{E}[X_v]\Big] \leq 2e^{\frac{-\mathbb{E}[X_v]}{3}} = 2e^{\frac{-25 \log d}{3}} \leq 2e^{-8 \log d} = 2d^{-8}.$$

Here we used Chernoff's bound (in the form of Remark 5.3 in the notes), according to which $\mathbb{P}\Big[|X - \mathbb{E}[X]| \geq \delta \mathbb{E}[X]\Big] \leq 2e^{-\delta^2 \mathbb{E}[X]/3}$, for every $0 \leq \delta \leq 1$, where X is a binomial random variable.

Note that A_v is mutually independent of the collection of events A_u for which $N(u) \cap N(v) = \emptyset$. Since G is d-regular, there are at most d^2 vertices u with $N(u) \cap N(v) \neq \emptyset$. It follows that A_v is mutually independent of a collection of all but at most d^2 events A_u.

We have

$$e \cdot \frac{2}{d^8} \cdot (d^2 + 1) \leq \frac{4e}{d^6} \leq 1.$$

(Using $d \geq 2$.) Thus, by the local lemma, $\mathbb{P}[\cap_{v \in V} \bar{A}_v] > 0$, as required.

Problem 3. Let $G = (V, E)$ be a graph with chromatic number $\chi(G)$.

(a) Let $\{U_1, U_2\}$ be a partition of V and denote by $H_i = G[U_i]$ the subgraph induced by U_i for $i \in \{1, 2\}$. Show that $\chi(H_1) + \chi(H_2) \geq \chi(G)$.

(b)* Suppose $\chi(G) = 1000$ and let $U \subseteq V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that:

$$\mathbb{P}[\chi(H) \leq 400] < \frac{1}{100}.$$

Solution.

(a) Properly colour $G[U_1]$ with $\chi(H_1)$ colours, and properly colour $G[U_2]$ with $\chi(H_2)$ *new* colours. We thus obtain a proper colouring of G with $\chi(H_1) + \chi(H_2)$ colours, which implies that $\chi(G) \leq \chi(H_1) + \chi(H_2)$, as required.

(b) First, note that when U is a uniformly random subset of V, then so is $U^c = V \setminus U$. Hence, we have

$$\mathbb{E}\left[\chi(G[U])\right] = \mathbb{E}\left[\chi(G[U^c])\right].$$

However, by part (a), $\chi(G[U]) + \chi(G[U^c]) \geq \chi(G) = 1000$, so taking expectations it follows that $\mathbb{E}[\chi(G[U])] \geq 500$. That is, $\mathbb{E}[\chi(H)] \geq 500$.

Let $\{V_1, \ldots, V_{1000}\}$ be a partition of V into independent sets (this partition exists because $\chi(G) = 1000$). Note that a uniformly random subset $U \subset V$ is obtained by choosing uniformly random subsets $U_i \subset V_i$ for every $1 \leq i \leq 1000$, and setting $U = \bigcup_{i=1}^{1000} U_i$. Since V_i is an independent set, if we have sets $U \subset V$ and $U' \subset V$ such that $U \cap V_j = U' \cap V_j$ for every $j \neq i$, then $|\chi(G[U]) - \chi(G[U'])| \leq 1$.

Hence, when we choose $U_1 \subset V_1, \ldots, U_{1000} \subset V_{1000}$, changing U_i for some fixed i changes $\chi(G[U]) = \chi(H)$ by at most 1. So $\chi(H)$ is 1-Lipschitz.

Thus, by the Azuma-Hoeffding inequality, and using the fact that $\mathbb{E}[\chi(H)] \geq 500$,

$$\mathbb{P}[\chi(H) \leq 400] \leq \mathbb{P}[\chi(H) - \mathbb{E}[\chi(H)] \leq -100] \leq e^{\frac{-100^2}{2\cdot 1000}} = e^{-5} < \frac{1}{100}.$$